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Convergence of Weighted Polynomial Approximations to Solutions of Partial Differential Equations with Quasianalytical Coefficients

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Sequences of polynomial functions which converge to solutions of partial differential equations with quasianalytical coefficients are constructed. Estimates of the degree of convergence are also given. © 1998 Academic Press

1. INTRODUCTION

0.1 G. Freud and P. Nevai began to develop a theory of weighted approximation for weights on $\mathbb{R}$ (see [1, 2, 8, 9]).

In [3], Babin constructed and investigated solutions of differential equations with analytical coefficients using methods of approximation theory. This leads to an idea of extending his results to wider types of infinitely differentiable functions. By a result due to Babin [4], a polynomial approximation $u(x)=\lim_{n\to\infty} P_n(A)\, f$ of the solution $u(x)$ of the differential equation $Au(x)=f(x)$, where A is a semi-bounded, self-adjoint operator, exists if and only if the coefficients of the operator are quasianalytical.

We will consider in this paper differential operators with coefficients which belong to Carleman classes of quasianalytical functions. We recall fundamental definitions and notation (see also [3–5]).

Consider the set of sequences of positive numbers with the rate of growth greater than or equal to $(ck)^k$ for some $c>0$. The sequences $\{a_k\}$ and $\{b_k\}$ are said to be equivalent if there are $c_1, c_2>0$ such that, for some number k_0,

$$a_k \leqslant (c_1)^k\, b_k, \qquad b_k \leqslant (c_2)^k\, a_k, \qquad \text{for} \quad k>k_0.$$

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Let us consider a sequence $\{M(k)\}$ satisfying, in addition, the following condition: there exists a constant c_M such that

$$\frac{M(k)}{k!}\,\frac{M(n)}{n!}\leqslant c_M\,\frac{M(k+n)}{(k+n)!}. \tag{0.1.1}$$

Given a sequence of the type defined above, let M denote its equivalence class. Let Ω be a closed bounded domain in $\mathbb{R}^N$ with a smooth boundary or $\Omega=\mathbb{T}^N$, the N-dimensional torus. We denote by $C(M)$ the set of infinitely differentiable functions on Ω such that for any $u(x)\in C(M)$ there exist numbers r and B such that for any $k\in\mathbb{N}$ we have

$$\max_{x\in\Omega,\,|\alpha|=k}|D^\alpha u(x)|\leqslant Br^kM(k), \tag{0.1.2}$$

where $x=(x_1,...,x_N)$, $a=(a_1,...,a_N)$ is a multi-index, $\alpha\in\mathbb{Z}^N_+$, $|a|=a_1+\cdots+a_N$, $D=(D_1,...,D_N)$, $D_i=\partial/\partial x_i$, and $D^\alpha=D_1^{\alpha_1}\cdots D_N^{\alpha_N}$. Condition (0.1.1) shows that the class $C(M)$ is closed under multiplication.

Let A be a self-adjoint differential operator of the second order

$$Au(x)=-\sum_{i,\,j=1}^{N}D_i(a_{ij}(x)\,D_ju(x))+a_{00}(x)\,u(x), \tag{0.1.3}$$

where $a_{ij}(x)\in C(M)$ and $a_{ij}(x)=\overline{a_{ij}(x)}$. In the case where $\Omega\subset\mathbb{R}^N$, for A to be self-adjoint, the condition

$$\sum_{i=1}^{N}v_i(x)\,a_{ij}(x)=0,\qquad j=1,...,N \tag{0.1.4}$$

must be satisfied for any $x\in\partial\Omega$ and normal vector $v(x)=(v_1(x),...,v_N(x))$.

We also assume that operator A is semi-bounded, that is,

$$(Au,u)\geqslant b\,\|u\|^2\qquad\text{for any function } u\in C(M). \tag{0.1.5}$$

Here $(\cdot,\cdot)$ and $\|\cdot\|$ are the scalar product and norm in $L_2(\Omega)$, and b is a positive constant that bounds below the spectrum of A.

We note that (as in [3]), A may be a degenerate elliptic operator.

0.2. Consider the Cauchy problems for the differential equations

$$Au(x)=f_0(x), \tag{0.2.1}$$

$$\begin{aligned}&\partial_t^2u(x,t)=-Au(x,t)+f_0(x),\qquad t>0\\&u(x,0)=f_1(x),\\&\partial_tu(x,0)=f_2(x),\end{aligned} \tag{0.2.2}$$

$$\begin{aligned} \partial_t u(x,t) &= -A(x,t) + f_0(x), \qquad t>0, \\ u(x,0) &= f_1(x). \end{aligned} \tag{0.2.3}$$

The functions $f_i(x)$ (as well as $a_{ij}(x)$) belong to the class $C(M)$.

In this paper, we construct for each of these problems a sequence of functions $P_n^i(t,\lambda)$, $n=1, ...,$ that are polynomial in λ, such that the solution $u(x,t)$ of any problem (0.2.1), (0.2.2), and (0.2.3) has the polynomial approximation

$$u(x,t) = \lim_{n\to\infty} \sum_{i=0}^{k} P_n^i(A,t)\, f_i(x), \tag{0.2.4}$$

where $k=0$, 2, or 1 for problems (0.2.1), (0.2.2), (0.2.3), respectively.

0.3. For any sequence $\{M(k)\}$, we introduce a sequence $\{a_k\}$ by the formula $a_k = \sqrt[k]{M(k)}$.

Definition 0.3.1. The class $C(M)$ is quasianalytical if the sequence $\{a_k\}$ has the properties

$$\lim_{k\to\infty} a_k = \infty, \tag{0.3.1}$$

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty, \tag{0.3.2}$$

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty. \tag{0.3.3}$$

We note that $\{a_k\}$ satisfies the conditions in Definition 0.3.1 if and only if the sequence $b_k = M(k+1)/M(k)$ does.

Following the ideas of [3], we introduce the functions

$$\begin{aligned} F_n(z) &= \prod_{k=1}^{n} \left(1 + \frac{z^2}{a_k^2}\right), \\ F_\infty(z) &= \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{a_k^2}\right), \end{aligned} \tag{0.3.4}$$

$$\begin{aligned} S_n(z) &= \prod_{k=1}^{n} \left(1 - \frac{iz}{a_k}\right)^2, \\ T_n(z) &= \frac{1}{2}(S_n(z) + S_n(-z)). \end{aligned} \tag{0.3.5}$$

We note that by virtue of (0.3.3) the function $F_\infty(z)$ is well-defined and is a uniform limit of the sequence $\{F_n(z)\}$ on compact sets.

We will use in the sequel the properties of the functions (0.3.4), (0.3.5) listed in the following proposition (see also [3]).

PROPOSITION 0.3.2. (1) $T_n(-z)=T_n(z)$.

(2) $S_n(-x)=\overline{S_n(x)}$ *and* $T_n(x)=\Re S_n(x)$.

(3) $\lim_{x\to\infty} \arg S_n(x)=-\pi n$.

(4) $\lim_{x\to-\infty} \arg S_n(x)=\pi n$.

(5) $\Delta \arg S_n(x)=-2\pi n$.

(6) $T_n(x)$ *has* $2n$ *real roots.*

(7) $F_n(x)=|S_n(x)|$.

(8) *The positive roots of* $T_n(x)$ *are the roots of the equation*

$$2\sum_{i=1}^{n} \operatorname{arc\,tan}(x/a_i)=(2s-1)\,\pi/2, \qquad \text{where} \quad s=1, 2, ..., n.$$

(9) $T_n(x)=0 \Leftrightarrow \Re S_n(x)=0 \Leftrightarrow \arg S_n(x)=\pi/2+k\pi$, $k=-n, -n+1, ..., (n-2), (n-1)$.

(10) *If* $z=ix$, $x\in\mathbb{R}$, *the values of* $S_n(z)$ *and* $T_n(z)$ *are positive real numbers.*

1. THE STATIONARY EQUATION: AN ESTIMATE OF THE CONVERGENCE RATE

1.0. We consider the Hilbert space $H=L_2(\Omega)$. Let $D(A)$ be the domain of definition of the operator A. We denote $R_0(A, f)=\sup\{R \mid f\in D(F_\infty(R\sqrt{A}))\}$. In the sequel, we will consider only those $f\in C(M)$ that satisfy $R_0(A, f)>0$.

We define a new norm in $D(A)$ by the formula $\|v\|_2=\|Av\|$. It is clear that $\|v\|_2\geqslant r^{-2}\|v\|$, where $r^2=b$ is the number that bounds the spectrum of A from below.

1.1. LEMMA 1.1.1. *Suppose* $0<R<R_0(A, f)$. *Then*

$$\|F_n(R\sqrt{A-bI})\,f\|\leqslant\|F_\infty(R\sqrt{A-bI})\,f\|. \tag{1.1.1}$$

Proof. Using the spectral decomposition of A and formula (0.3.4), we obtain

$$\begin{aligned}\|F_n(R\sqrt{A-bI})\,f\|^2 &= \int_b^\infty |F_n(R\sqrt{\lambda-b})|^2\,d(E_\lambda f, f)\\ &\leqslant \int_b^\infty |F_\infty(R\sqrt{\lambda-b})|^2\,d(E_\lambda f, f)\\ &= \|F_\infty(R\sqrt{A-bI})\,f\|^2.\end{aligned}$$

LEMMA 1.1.2. *Let $u(x)$ be the solution of* (0.2.1), *let* $0<R<R_0(A, f)$, *and let $P_n(t)$ be a polynomial of nth degree. Then*

$$\|u-P_n(A)\,f\|_2 \leqslant \sup_{t\geqslant b} \frac{|1-tP_n(t)|}{F_{n+1}(R\sqrt{t-b})}\,\|F_\infty(R\sqrt{A-bI})\,f\|. \tag{1.1.2}$$

Proof. We obtain, by using the spectral decomposition,

$$\begin{aligned}\|u-P_n(A)\,f\|_2^2 &= \|Au-AP_n(A)\,f\|^2 = \|f-AP_n(A)\,f\|^2\\ &= \int_b^\infty (1-tP_n(t))^2\,d(E_t f, f)\\ &= \int_b^\infty \frac{(1-tP_n(t))^2}{F_{n+1}^2(R\sqrt{t-b})}\,F_{n+1}^2(R\sqrt{t-b})\,d(E_t f, f)\\ &\leqslant \sup_{t\geqslant b}\frac{|1-tP_n(t)|^2}{F_{n+1}^2(R\sqrt{t-b})}\int_b^\infty F_{n+1}^2(R\sqrt{t-b})\,d(E_t f, f)\\ &\leqslant \sup_{t\geqslant b}\frac{|1-tP_n(t)|^2}{F_{n+1}^2(R\sqrt{t-b})}\,\|F_{n+1}(R\sqrt{A-bI})\,f\|^2\\ &\leqslant \sup_{t\geqslant b}\frac{|1-tP_n(t)|^2}{F_{n+1}^2(R\sqrt{t-b})}\,\|F_\infty(R\sqrt{A-bI})\,f\|^2.\end{aligned}$$

We will now look for a polynomial $P_n(t)$ such that $\sup_{t\geqslant b}(|1-tP_n(t)|/F_{n+1}(R\sqrt{t-b}))$ is minimal. Let us introduce new variables $t=b+x^2$ and $b=r^2$. Then, $\sup(|1-tP_n(t)|/F_{n+1}(R\sqrt{t-b}))=\sup_{x\geqslant 0}(|1-(r^2+x^2)\,P_n(r^2+x^2)|/F_{n+1}(Rx))$. Define $P_n(r^2+x^2)$ by the formula

$$1-(r^2+x^2)\,P_n(r^2+x^2)=\frac{T_{n+1}(Rx)}{T_{n+1}(iRr)}. \tag{1.1.3}$$

We claim that $P_n(r^2+x^2)$ is a polynomial. To prove this, note that $F(x)=1-(T_{n+1}(Rx)/T_{n+1}(iRr))$ is divisible by r^2+x^2, since $F(ir)=F(-ir)=0$. In addition, it follows from (1) in Proposition 0.3.2 that the function $F(x)$ is even.

LEMMA 1.1.3. *Let* $P_n(r^2+x^2)$ *be defined by* (1.1.3). *Then*

$$\sup_{x\geqslant 0}\frac{|1-(r^2+x^2)\,P_n(r^2+x^2)|}{F_{n+1}(Rx)}=\frac{1}{T_{n+1}(iRr)}. \tag{1.1.4}$$

Proof. It follows from (2) and (7) in Proposition 0.3.2 that for any real x we have

$$\left|\frac{T_{n+1}(Rx)}{F_{n+1}(Rx)}\right|\leqslant 1.$$

Therefore

$$\sup_{x\geqslant 0}\frac{|1-(r^2+x^2)\,P_n(r^2+x^2)|}{F_{n+1}(Rx)}=\sup_{x\geqslant 0}\left|\frac{T_{n+1}(Rx)}{F_{n+1}(Rx)}\right|\frac{1}{T_{n+1}(iRr)}$$
$$\leqslant\frac{1}{T_{n+1}(iRr)}.$$

The same properties (2) and (7) in Proposition 0.3.2 imply that $|T_{n+1}(Rx_i)/F_{n+1}(Rx_i)|=1$ for all $x_i\in\mathbb{R}$ such that $S_{n+1}(Rx_i)$ is real, i.e., the supremum in (1.1.4) is attained at the points x_i. By properties (3), (4), and (5) from the same proposition, the function $T_{n+1}(Rx)/F_{n+1}(Rx)$ takes alternately the values $+1$ and -1, at the $2n+1$ points x_i with $\arg S_{n+1}(Rx_i)=k\pi$, as well as at $\pm\infty$. It follows from a Chebyshev-type theorem [10, Chap. 2] that the function

$$\sup_{t\geqslant b}\frac{|1-tQ_n(t)|}{F_{n+1}(R\sqrt{t-b})}$$

takes the minimal possible value over all polynomials $Q_n(t)$ of nth degree in the case where $Q_n(t)=P_n(t)$.

Lemmas 1.1.2 and 1.1.3 imply the following estimate for the polynomial approximation of the solution of Eq. (0.2.1)

COROLLARY 1.1.4. *If $P_n(t)$ is the polynomial defined by* (1.1.3), *then*

$$\|u - P_n(A) f\|_2 \leqslant \frac{1}{T_{n+1}(iRr)} \|F_\infty(R\sqrt{A - bI}) f\|. \tag{1.1.5}$$

1.2. The following theorem gives an estimate for the polynomials $T_n(z)$.

THEOREM 1.2.1. *Let $z = x + iy$. Then there exist a real constant c, independent of x, y, and n, and a function $\Theta(y)$ such that*

$$\Theta(y) = \begin{cases} c & \text{if } |y| \geqslant 1, \\ c\,|y| & \text{if } |y| \leqslant 1, \end{cases}$$

and

$$|T_n(x+iy)| \geqslant \Theta(y)\,|S_n(x+iy)|\,(1+x^2)^{-1}.$$

Proof. Since $T_n(z)$ is an even function, one can assume without loss of generality that $y \geqslant 0$. The function $T_n(z)$ can be represented as $T_n(z) = 1/2 S_n(z)(1 + Q_n(z))$, where $Q_n(z) = \prod_{k=1}^n q_k(z)$, $q_k(z) = ((a_k + iz)/(a_k - iz))^2$. We now proceed to finding estimates for $1 + Q_n(z)$. We shall consider several cases.

Case 1. $|z| \geqslant 4 \sum_{k=1}^n a_k$.
In this case we have $|1 + Q_n(z)| \geqslant 1$ and

$$\begin{aligned} |\arg Q_n(z)| &\leqslant 2 \sum_{k=1}^n |\arg(1 + a_k/(iz)) - \arg(1 - a_k/(iz))| \\ &\leqslant 2 \sum_{k=1}^n (|\arg(1 + a_k/(iz))| + |\arg(1 - a_k/(iz))|) \\ &\leqslant 2 \sum_{k=1}^n \left(\pi/2 \frac{a_k}{|z|} + \pi/2 \frac{a_k}{|z|}\right) \leqslant \pi/2. \end{aligned}$$

Here, we used the following elementary

PROPOSITION 1. *If $|w| < 1$, then* $|\arg(1 + w)| \leqslant \arcsin(|w|) \leqslant \pi/2\,|w|$.

Let us estimate $|q_k(z)|$. We have

$$\begin{aligned} |q_k(z)| &= \frac{|a_k - y + ix|^2}{|a_k + y - ix|^2} = \frac{(a_k - y)^2 + x^2}{(a_k + y)^2 + x^2} \\ &= 1 - \frac{4a_k y}{(a_k + y)^2 + x^2}. \end{aligned}$$

For any k we have $|q_k(z)| \leqslant 1$, since $y \geqslant 0$, $a_k \geqslant 0$, and $4ab \leqslant (a+b)^2$ for all a, b. Hence the inequality $|Q_n(z)| \leqslant |q_k(z)|$ holds for any k, and

$$\begin{aligned} |1+Q_n(z)| &\geqslant 1-|Q_n(z)| \geqslant 1-|q_k(z)| \\ &\geqslant 1-\left(1-\frac{4a_k y}{(a_k+y)^2+x^2}\right)=\frac{4a_k y}{(a_k+y)^2+x^2} \\ &=\frac{1}{(1+x^2)}\,\frac{4a_k y}{((a_k+y)^2/(1+x^2)+x^2/(1+x^2))} \\ &\geqslant \frac{4a_k y}{1+x^2}\,\frac{1}{(a_k+y)^2+1}. \end{aligned} \tag{1.2.1}$$

The last inequality follows from the evident inequalities $(a_k+y)^2/(1+x^2) \leqslant (a_k+y)^2$ and $x^2/(1+x^2) \leqslant 1$.

Case 2. $0 \leqslant y \leqslant 1$.
Inequality (1.2.1) implies that

$$|1+Q_n(z)| \geqslant \frac{y}{(1+x^2)}\,\frac{4a_k}{(a_k+y)^2+1} \geqslant c_1\,\frac{y}{(1+x^2)},$$

where $c_1 = \max_k(4a_k/(a_k+y)^2+1)$. In particular, $c_1 \geqslant 4a_1/(a_1)^2+1$.

Case 3. $1 \leqslant y \leqslant a_n$.
There exists a number k with $a_{k-1} \leqslant y \leqslant a_k$. Then

$$\begin{aligned} |1+Q_n(z)| &\geqslant \frac{y}{(1+x^2)}\,\frac{4a_k}{(a_k+y)^2+1} \geqslant \frac{1}{(1+x^2)}\,\frac{4a_k a_{k-1}}{(2a_k)^2+1} \\ &\geqslant c_2\,\frac{1}{(1+x^2)}, \end{aligned}$$

where $c_2 = \min_k(4a_k a_{k-1}/(2a_k)^2+1) \geqslant 0$.

Case 4. $1 \leqslant a_n \leqslant y \leqslant 2a_n$.
Taking $k=n$ in (1.2.1), we get

$$\begin{aligned} |1+Q_n(z)| &\geqslant \frac{y}{(1+x^2)}\,\frac{4a_n}{(a_n+y)^2+1} \geqslant \frac{1}{(1+x^2)}\,\frac{4a_n^2}{9n_n^2+1} \\ &\geqslant 0.4\,\frac{1}{(1+x^2)}. \end{aligned}$$

Case 5. $1 \leqslant 2a_n \leqslant y \leqslant 4\sum_{k=1}^n a_k$.
Since $a_k \leqslant a_n$, $a_k/y \leqslant 0.5$. We also have $x/y \leqslant x$, since $y \geqslant 1$. Therefore

$$|q_k(z)| = 1 - \frac{4a_k y}{(a_k + y)^2 + (x)^2} = 1 - \frac{4a_k/y}{(a_k/y+1)^2 + (x/y)^2}$$
$$\leqslant 1 - \frac{4a_k/y}{9/4 + x^2}.$$

Let us now estimate $\ln |Q_n(z)|$.

$$\ln |Q_n(s)| = \ln \prod_{k=1}^n |q_k(z)| = \sum_{k=1}^n \ln |q_k(z)|$$
$$\leqslant \sum_{k=1}^n \ln\left(1 - \frac{4a_k/y}{9/4+x^2}\right) \leqslant -\sum_{k=1}^n \left(\frac{4a_k/y}{9/4+x^2}\right)$$
$$\leqslant -\frac{4/y}{9/4+x^2}\sum_{k=1}^n a_k \leqslant -\frac{1}{9/4+x^2}. \tag{1.2.2}$$

For estimating $\ln(1-((4a_k/y)/(9/4+x^2)))$ we used the well-known

PROPOSITION 2. $\ln(1-\alpha) \leqslant -\alpha$ *for* $0 \leqslant \alpha < 1$.

It is clear that $\alpha = (4a_k/y)/(9/4+x^2) \leqslant 2/(9/4+x^2) < 1$. The condition $y \geqslant 2\sum_{k=1}^n a_k$ was employed in the last inequality. It follows from (1.2.2) that

$$|1 + Q_n(z)| \geqslant 1 - |Q_n(z)| = 1 - \exp\{\ln |Q_n(z)|\}$$
$$\geqslant 1 - \exp\left(-\frac{1}{9/4+x^2}\right). \tag{1.2.3}$$

We will now use the inequality following

PROPOSITION 3. $1 - \exp(-\alpha) \geqslant (1/e)\,\alpha$ *for* $0 \leqslant \alpha < 1$.

This proposition and (1.2.3) imply that

$$|1 + Q_n(z)|\ 1 - \exp\left(-\frac{1}{9/4+x^2}\right) \geqslant 1/e\,\frac{1}{9/4+x^2}$$
$$\geqslant 4/(9e)\,\frac{1}{1+x^2}.$$

This ends the proof of Theorem 1.2.1.

We denote $\prod_{k=m}^{n}(1-(iz/a_k))^2$ by $S_n^m(z)$.

It is clear that $S_n(z)=S_n^m(z)\,S_{m-1}(z)$. For every m there exists a constant c such that

$$|S_{m-1}(z)|\geqslant c(1+x^2)^{m-1}.$$

This yields

PROPOSITION 1.2.2. *Under the hypotheses of Theorem* 1.2.1 *we have*

$$|T_n(x+iy)|\geqslant \Theta(y)\,|S_n^m(x+iy)|\,(1+x^2)^{m-2}.$$

COROLLARY 1.2.3. *The polynomials $P_n(\lambda)$ defined by* (1.1.3) *satisfy the estimate*

$$\|u(x)-P_n(A)\,f(x)\|_2\leqslant\frac{1}{\Theta(Rr)}\,\frac{1}{|S_n(iRr)|}\,\|f_\infty(R\sqrt{A-bI})\,f\|.$$

1.3. We give an estimate for the functions $|S_n(iRr)|$.

LEMMA 1.3.1. *For any sequence $\{a_k\}$ satisfying* (0.3.1)–(0.3.3) *there exists a constant c such that*

$$|S_n(iRr)|\geqslant c\exp\left(2Rr\sum_{k=1}^{n}1/a_k\right).$$

Proof. We will use the fact that $\ln(1+x)\geqslant x-x^2/2$ for $0<x\leqslant 1$. By virtue of (0.3.1), $a_k\to\infty$ as $k\to\infty$. Therefore, for some n_0, we have $Rr/a_k<1$ for $n\geqslant n_0$. In addition, $\sum_{k=1}^{\infty}1/a_k^2<\infty$ by (0.3.3). Then

$$\begin{aligned}
\ln|S_n(iRr)| &= \ln\left|\prod_{k=1}^{n}\left(1+\frac{Rr}{a_k}\right)^2\right| = 2\sum_{k=1}^{n}\ln\left(1+\frac{Rr}{a_k}\right)\\
&= 2\sum_{k=1}^{n_0}\ln\left(1+\frac{Rr}{a_k}\right)+2\sum_{k=n_0}^{n}\ln\left(1+\frac{Rr}{a_k}\right)\\
&\geqslant 2\sum_{k=1}^{n_0}\ln\left(1+\frac{Rr}{a_k}\right)+2\sum_{k=n_0}^{n}\left(\frac{Rr}{a_k}-\frac{R^2r^2}{2a_k^2}\right)\\
&= 2\sum_{k=1}^{n_0}\ln\left(1+\frac{Rr}{a_k}\right)-2\sum_{k=1}^{n_0}\left(\frac{Rr}{a_k}\right)\\
&\quad -R^2r^2\sum_{k=n_0}^{n}(1/a_k^2)+2\sum_{k=1}^{n}\frac{Rr}{a_k}
\end{aligned}$$

$$\begin{aligned}&\geqslant 2\sum_{k=1}^{n_0}\ln\left(1+\frac{Rr}{a_k}\right)-2\sum_{k=1}^{n_0}\left(\frac{Rr}{a_k}\right)\\&\quad-R^2r^2\sum_{k=n_0}^{\infty}(1/a_k^2)+2\sum_{k=1}^{n}\frac{Rr}{a_k}\\&=c_1+2Rr\sum_{k=1}^{n}\frac{1}{a_k}.\end{aligned}$$

Here,

$$c_1=2\sum_{k=1}^{n_0}\ln\left(1+\frac{Rr}{a_k}\right)-2\sum_{k=1}^{n_0}\left(\frac{Rr}{a_k}\right)-R^2r^2\sum_{k=1}^{\infty}(1/a_k^2).$$

Hence

$$\begin{aligned}|S_n(iRr)|&=\exp\{\ln|S_n(iRr)|\}\geqslant\exp\left\{c_1+2Rr\sum_{k=1}^{n}1/a_k\right\}\\&=c\exp\left\{2Rr\sum_{k=1}^{n}1/a_k\right\}.\end{aligned}$$

The following estimate can be obtained in a similar way.

PROPOSITION 1.3.2. $|S_n(iRr)|\geqslant c\exp(2Rr\sum_{k=m}^{n}1/a_k)$.

Lemma 1.3.1 and Corollary 1.2.3 imply

THEOREM 1.3.3. *Let* $0<R<R_0(A,f)$ *and let* $P_n(\lambda)$ *be defined by* (1.1.3). *Then*

$$\|u(x)-P_n(A)f(x)\|\leqslant c(R,r)\exp\left\{-2Rr\sum_{k=1}^{n}1/a_k\right\}\|F_\infty(R\sqrt{A-bI})f\|.$$

1.4. Let us consider the following examples.

1.4.1. Let $C(M)$ be the class of analytic functions (see [3]). Then $M(k)=k^k$, and $a_k=k$. The sequence $\{a_k\}$ satisfies all properties (0.3.1)–(0.3.3), and $\sum_{k=1}^{n}1/k=\ln(n)+\gamma+f(n)/n$, where γ is the Euler constant and $0<f(n)<1$. Theorem 1.3.2 gives us the following estimate of the convergence rate of the polynomial approximations $P_n(A)f(x)$ to the solution of (0.2.1):

$$\|u(x)-P_n(A)f(x)\|\leqslant c(R,r)n^{-2Rr}\|F_\infty(R\sqrt{A-bI})f\|.$$

Here $F_\infty(z)=\prod_{k=1}^{\infty}(1+z^2/k^2)=\sinh(\pi z)/\pi z$.

1.4.2. Let $C(M)$ be the Denjoy quasianalytical class. This means that

$$M(k) = (\underbrace{k \cdot \ln k \cdot \ln \ln k \cdot \cdots \cdot \ln \cdots \ln k}_{s \text{ factors}})^k.$$

The sequence $\{a_k\}$ may be chosen by the formula $a_k = k \cdot \ln k \cdot (\ln \ln k) \cdot \cdots \cdot (\ln \cdots \ln k)$. Properties (0.3.1)–(0.3.3) hold, and for $\sum_{k=1}^{n} 1/a_k = \ln \ln \cdots \ln k + f(n)$ exist constants c_1 and c_2 such that $c_1 < f(n) < c_2$. Theorem 1.3.2 yields the following estimate of the convergence rate of the polynomial approximation $P_n(A)\, f(x)$ to the solution of (0.2.1):

$$\|u(x) - P_n(A)\, f(x)\| \leqslant c(R, r)(\ln \ln \cdots \ln(n))^{-2Rr}\, \|F_\infty(R\sqrt{A - bI})\, f\|.$$

1.4.3. Let $C(M)$ be the quasianalytical class defined by the sequence $M(k) = (k \ln^\nu k)^k$, where $0 < \nu < 1$. Choose the sequence $a_k = k \ln^\nu k$. Properties (0.3.1)–(0.3.3) hold, and for $\sum_{k=1}^{n} 1/a_k = 1/(1-\nu) \ln^{1-\nu}(n) + f(n)$, there exist constants c_1 and c_2 such that $c_1 < f(n) < c_2$. Theorem 1.3.2 yields the following estimate of the convergence rate of the polynomial approximation $P_n(A)\, f(x)$ to the solution of (0.2.1):

$$\|u(x) - P(A)\, f(x)\| \leqslant c(R, r) \exp\left\{ -\frac{2Rr}{1-\nu} \ln^{1-\nu}(n) \right\} \|F_\infty(R\sqrt{A - bI})\, f\|.$$

2. THE HYPERBOLIC EQUATION: AN ESTIMATE OF THE CONVERGENCE RATE

2.0. In this section the hyperbolic equation (0.2.2) is studied. The solution of (0.2.2) is given by the formula

$$u(x, t) = G_0(t, A)\, f_0(x) + G_1(t, A)\, f_1(x) + G_2(t, A)\, f_2(x), \qquad (2.0.1)$$

where

$$\begin{aligned} G_0(t, \lambda) &= \frac{(1 - \cos(t\sqrt{\lambda}))}{\lambda}, \\ G_1(t, \lambda) &= \cos(t\sqrt{\lambda}), \qquad (2.0.2) \\ G_2(t, \lambda) &= \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}. \end{aligned}$$

We introduce new variables

$$\lambda = z^2, \qquad H_i(t, z) = G_i(t, z^2). \tag{2.0.3}$$

We denote $J_\beta = \{z \in \mathbb{C} \mid |\Im z| \leqslant \beta\}$.

DEFINITION 2.0.1. For any $q > 0$ let

$$\begin{aligned}\mathscr{U}_q(J_\beta) = \{f(z) \in \mathcal{O}(J_\beta) \mid |f(x+iy)| \leqslant c(1+|x|)^q,\\ z = x + iy \in J_\beta\}\end{aligned}$$

be the class of analytic functions in J_β whose rate of the growth does not exceed the qth degree of $x = \Re(z)$.

We denote $\mathscr{U}(J_\beta) = \bigcup_{q>0} \mathscr{U}_q(J_\beta)$.

The functions $H_i(t, z)$ belong to the class $\mathscr{U}_q(J_\beta)$ for a fixed t and are even in z. Let us construct a sequence $\{P_n^i(t, z)\}$ of even polynomials $P_n^i(t, z)$, $n = 1, 2, \ldots$, of degree $2n-2$ such that the sequence $\sum_{i=0}^2 P_n^i(t, \sqrt{A}) f_i(x)$ converges to $u(x, t)$ in the norm of the space H. For any i, define $P_n^i(t, z)$ as the interpolation polynomial of $H_i(t, z)$ that takes the same values as $H_i(t, z)$ at the zeros of $T_n(z)$. Since the functions $H_i(t, z)$ and $T_n(z)$ are even in z, the polynomial $P_n^i(t, z)$ is even in z.

2.1. The following two propositions hold for any function $H(z) \in \mathscr{U}_q(J_\beta)$.

2.1.1. *The Hermite formula* [7]. Let $P_n(z)$ be the interpolation polynomial of the function $H(z)$ constructed on the basis of the zeros of $T_n(z)$, that is, taking the values of $H(z)$ at $z \in \{z_i \mid T_n(z_i) = 0,\ i = 1, \ldots, 2n\}$. Then

$$H(z) - P_n(z) = \frac{1}{2\pi} T_n(z) \int_\gamma \frac{H(\zeta)}{T_n(\zeta)} \frac{d\zeta}{\zeta - z}, \tag{2.1.1}$$

where the contour γ bounds the zeros of $T_n(z)$ and the point z. We take as γ the two lines parallel to the x axis (that contains the zeros of the function $T_n(z)$).

2.1.2. *Spectral decomposition formula.* Any function $H(z)$ in $\mathscr{U}_q(J_\beta)$ satisfies the spectral decomposition formula

$$\|H(A) f\|^2 = \int_b^\infty |H(\mu)|^2 \, d(E_\mu f, f), \tag{2.1.2}$$

where as earlier, A is a self-adjoint semi-bounded operator, b bounds the spectrum of A from below, and $f \in D(A)$.

2.2. We now present the main result of this section.

THEOREM 2.2.1. *Let $u(x, t)$ be the solution of* (0.2.2), *where A is a self-adjoint semi-bounded operator with coefficients from the class $C(M)$, and let $f_i(x)$ belong to $C(M)$. Let $P_n^i(t, z)$, $n=1, 2, ...$, be the interpolation polynomials of the functions $H_i(t, z)$ defined by* (2.0.2) *and* (2.0.3) *that take the same values as the functions $H_i(t, z)$ at the zeros of the polynomials $T_n(z)$. The class $C(M)$ is defined by the sequence $\{a_n\}$, where $n \in \mathbb{N}$; the sequence $\{a_n\}$ is defined by a monotonously increasing function $a(x)$ (that is, $a_n = a(n)$) having the following properties*:

(1) *$x/a(x)$ is a monotonous non-increasing function,*

(2) *$x/\exp\{(t/2R)(a(x)/x)\}$ is an increasing function that tends to infinity as $x \to \infty$.*

Let $f_i(x) \in D(F_\infty(R\sqrt{A}))$ for all i. Then there exist positive constants c_1 and c_2 such that

$$\left\| u(x, t) - \sum_{i=0}^{2} P_n^i(t, \sqrt{A})\, f_i(x) \right\| \leqslant c_1 \exp\left\{ -2c_2 \frac{n}{\exp((t/2R)(a_n/n))} \right\}.$$

The constant c_2 is close to 1.

Remark. The classes of quasianalytic functions determined by the hypotheses of the theorem are non-void: these hypotheses are satisfied by the classes described in Examples 1.4.1 and 1.4.3.

Proof. It follows from (2.0.1) and (2.1.2) that

$$\begin{aligned}
&\left\| u(x, t) - \sum_{i=0}^{2} P_n^i(t, \sqrt{A})\, f_i(x) \right\|^2 \\
&\quad = \left\| \sum_{i=0}^{2} G_i(t, A)\, f_i(x) - \sum_{i=0}^{2} P_n^i(t, \sqrt{A})\, f_i(x) \right\|^2 \\
&\quad \leqslant \sum_{i=0}^{2} \| H_i(t, \sqrt{A})\, f_i(x) - P_n^i(t, \sqrt{A})\, f_i(x) \|^2 \\
&\quad = \sum_{i=0}^{2} \int_b^\infty |H_i(t, \sqrt{\lambda}) - P_n^i(t, \sqrt{\lambda})|^2\, d(E_\lambda f, f). \qquad (2.2.1)
\end{aligned}$$

Since all $H_i(t, z)$ belong to the same class $\mathcal{U}_q(J_\beta)$, we may consider one summand only. Denote $H_i(t, z)$ by $H(t, z)$, $P_n^i(t, z)$ by $P_n(t, z)$, $f_i(x)$ by $f(x)$, respectively. Then

$$\int_b^\infty |H(t,\sqrt{\lambda})-P_n(t,\sqrt{\lambda})|^2\, d(E_\lambda f, f)$$

$$=\int_b^\infty \frac{|H(t,\sqrt{\lambda})-P_n(t,\sqrt{\lambda})|^2}{F_\infty^2(R\sqrt{\lambda})} F_\infty^2(R\sqrt{\lambda})\, d(E_\lambda f, f)$$

$$\leqslant \max_{\lambda\geqslant b} \frac{|H(t,\sqrt{\lambda})-P_n(t,\sqrt{\lambda})|^2}{F_\infty^2(R\sqrt{\lambda})} \int_b^\infty F_\infty^2(R\sqrt{\lambda})\, d(E_\lambda f, f)$$

$$=\max_{\lambda\geqslant b} \frac{|H(t,\sqrt{\lambda})-P_n(t,\sqrt{\lambda})|^2}{F_\infty^2(R\sqrt{\lambda})} \|F_\infty(R\sqrt{A})\, f(x)\|^2. \tag{2.2.2}$$

One can deduce from (2.2.1) and (2.2.2) that

$$\left\| u(x,t)-\sum_{i=0}^{2} P_n^i(t,\sqrt{A})\, f_i(x)\right\| \leqslant c_1 \max_{\lambda\geqslant b} \frac{|H(t,\sqrt{\lambda})-P_n(t,\sqrt{\lambda})|}{F_\infty(R\sqrt{\lambda})}, \tag{2.2.3}$$

where $c_1=3c\,\|F_\infty(R\sqrt{A})\, f(x)\|$.

To estimate $\max_{\lambda\geqslant b}(|H(t,\sqrt{\lambda})-P_n(t,\sqrt{\lambda})|/F_\infty(R\sqrt{\lambda})$, we will substitute λ for z^2 and apply formula (2.1.1) to the polynomial $T_n(Rz)$. Then

$$|H(t,z)-P_n(t,z)|$$

$$\leqslant \frac{1}{2\pi}|T_n(Rz)| \left| \int_{\partial J_\beta} \frac{H(t,\zeta)}{T_n(R\zeta)} \frac{d\zeta}{\zeta-z}\right|$$

$$\leqslant \frac{1}{2\pi}|T_n(Rz)| \left| \int_{-\infty,\, \Im z=-\beta}^{\infty} \frac{H(t,\zeta)}{T_n(R\zeta)} \left(\frac{1}{\zeta-z}-\frac{1}{\zeta+z}\right) d\zeta\right|$$

$$\leqslant \frac{1}{2\pi}|T_n(Rz)| \sup_{\zeta\in\partial J_\beta} \left|\frac{H(t,\zeta)}{T_n(R\zeta)}\right| \int_{-\infty,\, \Im\zeta=-\beta}^{\infty} \left|\frac{1}{\zeta-z}-\frac{1}{\zeta+z}\right| d\zeta.$$

Using (2.1.3), (2.1.4), and (0.3.5), we finally obtain

$$\left\| u(x,t)-\sum_{i=0}^{2} P_n^i(t,\sqrt{A})\, f_i(x)\right\| \leqslant c \sup_{\zeta\in\partial J_\beta} \left|\frac{H(t,\zeta)}{T_n(R\zeta)}\right| \max_{x\geqslant b} \frac{T_n(Rx)}{F_\infty(Rx)}$$

$$\leqslant c \sup_{\zeta\in\partial J_\beta} \left|\frac{H(t,\zeta)}{T_n(R\zeta)}\right|. \tag{2.2.4}$$

Let us now estimate $\sup_{\zeta\in\partial J_\beta}|H(t,\zeta)/T_n(R\zeta)|$ for various values of β. Let us find $\min_y \sup_{\zeta\in\partial J_\beta}|H(t,\zeta)/T_n(R\zeta)|$.

LEMMA 2.2.2. *Under the hypotheses of Theorem* 2.2.1,

$$\min_{y} \sup_{\zeta = x+iy} \left| \frac{H(t,\zeta)}{T_n(R\zeta)} \right| \leqslant c_1 \exp\left\{ -2c_2 \frac{n}{\exp((t/2R)(a_n/n))} \right\}.$$

Proof. For all functions $H_i(t,z)$ the estimate $|H_i(t,z)| \leqslant c_2 \exp(ty)$, $y = \Im z$, holds. Moreover, Theorem 1.2.1 implies that

$$|T_n(x+iy)| \geqslant \Theta(y)\, |S_n^m(x+iy)|\, (1+x^2)^{m-2},$$

where

$$\Theta(y) = \begin{cases} c & \text{if} \quad |y| \geqslant 1, \\ c\,|y| & \text{if} \quad |y| \leqslant 1 \end{cases}.$$

Hence

$$\min_{y} \sup_{\zeta = x+iy} \left| \frac{H(t,\zeta)}{T_n(R\zeta)} \right| \leqslant c \min_{y} \sup_{x} \frac{\exp(ty)}{\Theta(y)\, |S_n^m(Rx+iRy)|\, (1+x^2)^{m-2}}.$$

By substituting in the last inequality Rx and Ry for x and y, respectively, and 2τ for t/R, we get

$$\begin{aligned} \min_{y \geqslant 1} \sup_{\zeta = x+iy} \left| \frac{H(t,\zeta)}{T_n(R\zeta)} \right| &\leqslant c_3 \min_{y \geqslant R} \sup_{x} \exp\{2\tau y - \ln(|S_n^m(x+iy)|)\} \\ &= c_3 \exp\left\{ \min_{y} \sup_{x} \left(2\tau y - 2\sum_{k=m}^{n} \ln\left| 1 + \frac{|y| - ix}{a_k} \right| \right) \right\} \\ &\leqslant c_3 \exp\left\{ \min_{y \geqslant R} \left(2\tau y - 2\sum_{k=m}^{n} \ln\left(1 + \frac{y}{a_k} \right) \right) \right\}. \qquad (2.2.5) \end{aligned}$$

We now estimate $\tau y - \sum_{k=m}^{n} \ln(1 + y/a_k)$ using the Euler–Maclaurin formula (see [6]):

$$\begin{aligned} \tau y - \sum_{k=m}^{n} \ln\left(1 + \frac{y}{a_k}\right) &= \tau y - \int_m^n \ln\left(1 + \frac{y}{a(\zeta)}\right) d\zeta \\ &\quad + \frac{1}{2}\left(\ln\left(1 + \frac{y}{a_n}\right) - \ln\left(1 + \frac{y}{a_m}\right) \right) + O(1). \end{aligned}$$

By integrating the last equality by parts and using the fact that $x/a(x)$ is a non-increasing function (so that $x/a(x) \geqslant n/a_n$ for $x \leqslant n$) we have, further

$$\tau y - \lambda \ln(1 + y/a(\lambda))|_{\lambda=m}^{\lambda=n} - y \int_m^n \frac{\lambda}{a(\lambda)} \frac{a'(\lambda)}{y + a(\lambda)}\, d\lambda$$

$$+ \frac{1}{2}(\ln(1 + y/a_n) - \ln(1 + y/a_m)) + O(1)$$

$$\leqslant \tau y - n \ln(1 + y/a_n) + m \ln(1 + m/a_m) - y \frac{n}{a_n} \int_m^n \frac{a'(\lambda)}{y + a(\lambda)}\, d\lambda$$

$$+ \frac{1}{2}(\ln(1 + y/a_n) - \ln(1 + y/a_m)) + O(1)$$

$$\leqslant \tau y - n \ln(1 + y/a_n) + m \ln(1 + y/a_m) - y \frac{n}{a_n} \ln(y + a(\lambda))\Big|_{\lambda=m}^{\lambda=n}$$

$$+ \frac{1}{2}(\ln(1 + y/a_n) - \ln(1 + y/a_m)) + O(1)$$

$$\leqslant \tau y - \left(n - \frac{1}{2}\right) \ln(1 + y/a_n) + \left(m - \frac{1}{2}\right) \ln(1 + y/a_m)$$

$$- y \frac{n}{a_n} \ln\left(\frac{y + a_n}{y + a_m}\right) + c_4.$$

Denoting the last expression by $I(y)$, we obtain that

$$\min_{y \geqslant R} \left\{ \tau y - \sum_{k=m}^{n} \ln(1 + y/a_k) \right\} \leqslant \min_{y \geqslant R} I(y).$$

For any function $\min_{x \in \Omega} f(x) \leqslant f(x_0)$ for $x_0 \in \Omega$, let us take $y = y_0(n) = (a_n - a_m \exp(\tau a_n/n))/\exp(\tau a_n/n) - 1$. By condition (2) of Theorem 2.2.1, $y_0(n) \to \infty$ as $n \to \infty$. Therefore, beginning with some n, we have $y_0(n) > R$. Hence

$$\min_{y \geqslant R} \left\{ \tau y - \sum_{k=m}^{n} \ln(1 + y/a_k) \right\} \leqslant I(y_0(n)).$$

By substituting $y_0(n)$ into $I(y)$ and simplifying it, we get

$$I(y_0(n)) = -n \ln(1 + y_0(n)/a_n) \left(1 - \frac{1}{2n} + \frac{(m - 1/2) \ln(1 + y_0(n)/a_m)}{n \ln(1 + y_0(n)/a_n)}\right).$$

The following lemma is obvious.

LEMMA 2.2.3. *Under the hypotheses of Theorem* 2.2.1, *we have*

(1) $\lim_{n\to\infty} ((m-1/2)\ln(1+y_0(n)/a_m)/n\ln(1+y_0(n)/a_n))=0$,

(2) $\lim_{n\to\infty} (\ln(1+y_0(n)/a_n)/\exp(-\tau a_n/n))=1$.

Lemma 2.2.3 implies that there exists a constant c_2, sufficiently close to 1, such that for all n large enough we have

$$\min_{y\geqslant R} I(y)\leqslant -2c_2 n\exp(-\tau a_n/n). \tag{2.2.6}$$

By substituting (2.2.6) into (2.2.5) and taking into account all the changes of variables, we finally obtain that

$$\left\| u(x,t)-\sum_{i=0}^{2} P_n^i(t,\sqrt{A})\, f_i(x)\right\| \leqslant c_1 \exp\left\{-2c_2\frac{n}{\exp((t/2R)(a_n/n))}\right\}.$$

The theorem is proved.

3. THE PARABOLIC EQUATION: AN ESTIMATE OF THE CONVERGENCE RATE

3.0. In this section the hyperbolic equation (0.2.3) is studied. The solution of (0.2.3) is given by the formula

$$u(x,t)=G_0(t,A)\, f_0(x)+G_1(t,A)\, f_1(x), \tag{3.0.1}$$

where

$$\begin{aligned} G_0(t,\lambda)&=\frac{(1-\exp(-t\lambda))}{\lambda},\\ G_1(t,\lambda)&=\exp(-t\lambda). \end{aligned} \tag{3.0.2}$$

3.1. We shall assume throughout this section that the notation and the changes of variables introduced in Section 2 are in force.

3.2. We now present the main result of this section.

THEOREM 3.2.1. *Let* $u(x,t)$ *be the solution of* (0.2.3), *where* A *is a self-adjoint semi-bounded operator with coefficients from the class* $C(M)$, *and let* $f_i(x)$ *belong to* $C(M)$. *Let* $P_n^i(t,z)$, $n=1,2,\ldots$, *be the interpolation polynomials of the functions* $H_i(t,z)$ *defined by* (3.0.2) *and* (2.0.3) *that take the same values as the functions* $H_i(t,z)$ *at the zeros of the polynomials* $T_n(z)$. *The class* $C(M)$ *is defined by the sequence* $\{a_n\}$, *where* $n\in\mathbb{N}$; *the sequence*

$\{a_n\}$ is defined by a monotonous increasing function $a(x)$ (that is, $a_n = a(n)$) having the following properties:

(1) *$x/a(x)$ is a monotonous non-increasing function,*

(2) *$(x \ln a(x))/a(x)$ is an increasing function that tends to infinity as $x \to \infty$.*

Let $f_i(x) \in D(F_\infty(R\sqrt{A}))$ for all i. Then there exist positive constants c_1 and c_2 such that

$$\left\| u(x,t) - \sum_{i=0}^{1} P_n^i(t, \sqrt{A})\, f_i(x) \right\| \leqslant c_1 \exp\left\{ -\frac{c_2 R^2}{t} \left(\frac{n \ln a_n}{a_n} \right)^2 \right\}.$$

The constant c_2 is close to 1.

Proof. As in the proof of Theorem 2.2.1, it follows from (2.1.1), (2.1.2), (3.0.1), and (3.0.2) that

$$\left\| u(x,t) - \sum_{i=0}^{1} P_n^i(t, \sqrt{A})\, f_i(x) \right\| \leqslant c \sup_{\zeta \in \partial J_\beta} \left| \frac{H(t,\zeta)}{T_n(R\zeta)} \right| \max_{x \geqslant b} \frac{T_n(Rx)}{F_\infty(Rx)}$$
$$\leqslant c \sup_{\zeta \in \partial J_\beta} \left| \frac{H(t,\zeta)}{T_n(R\zeta)} \right|. \tag{3.2.1}$$

Lemma 3.2.2. *Under the hypotheses of Theorem* 3.2.1,

$$\min_{y} \sup_{\zeta = x+iy} \left| \frac{H(t,\zeta)}{T_n(R\zeta)} \right| \leqslant c_1 \exp\left\{ -\frac{c_2 R^2}{t} \left(\frac{n \ln a_n}{a_n} \right)^2 \right\}.$$

Proof. For all functions $H_i(t,z)$ the estimate $|H_i(t,z)| \leqslant c_2 \exp(ty^2)$, $y = \Im z$, holds. Moreover, Theorem 1.2.1 implies that

$$|T_n(x+iy)| \geqslant \Theta(y)\, |S_n^m(x+iy)|\, (1+x^2)^{m-2},$$

where

$$\Theta(y) = \begin{cases} c & \text{if } \ |y| \geqslant 1, \\ c\,|y| & \text{if } \ |y| \leqslant 1. \end{cases}$$

Hence

$$\min_{y} \sup_{\zeta = x+iy} \left| \frac{H(t,\zeta)}{T_n(R\zeta)} \right| \leqslant c \min_{y} \sup_{x} \frac{\exp(ty^2)}{\Theta(y)\, |S_n^m(Rx + iRy)|\, (1+x^2)^{m-2}}.$$

By substituting Rx and Ry for x and y, respectively, and 2τ for $t/(R^2)$, we get

$$\begin{aligned}\min_{y\geqslant 1}\ \sup_{\zeta=x+iy}\left|\frac{H(t,\zeta)}{T_n(R\zeta)}\right| &\leqslant c_3 \min_{y\geqslant R}\ \sup_{x} \exp\{\tau y^2-\ln(|S_n^m(x+iy)|)\}\\ &= c_3 \exp\left\{\min_{y}\ \sup_{x}\left(\tau y^2-2\sum_{k=m}^{n}\ln\left|1+\frac{|y|-ix}{a_k}\right|\right)\right\}\\ &\leqslant c_3 \exp\left\{\min_{y\geqslant R} 2\left(\frac{\tau}{2}y^2-\sum_{k=m}^{n}\ln\left(1+\frac{y}{a_k}\right)\right)\right\}. \qquad (3.2.2)\end{aligned}$$

Now estimate $(\tau/2)\, y^2-\sum_{k=m}^{n}\ln(1+y/a_k)$. By integrating the last equality by parts and using the fact that $x/a(x)$ is a non-increasing function (so that $x/a(x)\geqslant n/a_n$ for $x\leqslant n$), we have

$$\begin{aligned}&\frac{\tau}{2}y^2-\sum_{k=m}^{n}\ln\left(1+\frac{y}{a_k}\right)\\ &\quad=\frac{\tau}{2}y^2-\int_m^n\ln\left(1+\frac{y}{a(\zeta)}\right)d\zeta+\frac{1}{2}\left(\ln\left(1+\frac{y}{a_n}\right)-\ln\left(1+\frac{y}{a_m}\right)\right)+O(1)\\ &\quad=\frac{\tau}{2}y^2-\lambda\ln\left(1+\frac{y}{a(\lambda)}\right)\Big|_{\lambda=m}^{\lambda=n}-y\int_m^n\frac{\lambda}{a(\lambda)}\frac{a'(\lambda)}{y+a(\lambda)}\,d\lambda\\ &\qquad+\frac{1}{2}\left(\ln\left(1+\frac{y}{a_n}\right)-\ln\left(1+\frac{y}{a_m}\right)\right)+O(1)\\ &\quad\leqslant\frac{\tau}{2}y^2-n\ln(1+y/a_n)+m\ln(1+m/a_m)-y\frac{n}{a_n}\int_m^n\frac{a'(\lambda)}{y+a(\lambda)}\,d\lambda\\ &\qquad+\frac{1}{2}\left(\ln\left(1+\frac{y}{a_n}\right)-\ln\left(1+\frac{y}{a_m}\right)\right)+O(1)\\ &\quad\leqslant\frac{\tau}{2}y^2-n\ln(1+y/a_n)+m\ln(1+y/a_m)-y\frac{n}{a_n}\ln(y+a(\lambda))\Big|_{\lambda=m}^{\lambda=n}\\ &\qquad+\frac{1}{2}\left(\ln\left(1+\frac{y}{a_n}\right)-\ln\left(1+\frac{y}{a_m}\right)\right)+O(1)\\ &\quad\leqslant\frac{\tau}{2}y^2-\left(n-\frac{1}{2}\right)\ln\left(1+\frac{y}{a_n}\right)+\left(m-\frac{1}{2}\right)\ln\left(1+\frac{y}{a_m}\right)\\ &\qquad-y\frac{n}{a_n}\ln\left(\frac{y+a_n}{y+a_m}\right)+c_4.\end{aligned}$$

Denoting the last expression by $I(y)$, we obtain that

$$\min_{y \geqslant R} \left\{ \frac{\tau}{2} y^2 - \sum_{k=m}^{n} \ln(1 + y/a_k) \right\} \leqslant \min_{y \geqslant R} I(y).$$

Let us take $y = y_1(n) = (1/\tau)(n \ln a_n/a_n)$. By condition (2) of Theorem 3.2.1, $y_1(n) \to \infty$ as $n \to \infty$. Therefore, beginning with some n, we have $y_1(n) > R$. Hence

$$\min_{y \geqslant R} \left\{ \frac{\tau}{2} y^2 - \sum_{k=m}^{n} \ln(1 + y/a_k) \right\} \leqslant I(y_1(n)). \tag{3.2.3}$$

By substituting $y_1(n)$ into $I(y)$ and simplifying it, we get

$$I(y_1(n)) = -\frac{1}{2\tau} \left(\frac{n \ln a_n}{a_n} \right)^2 \left(1 + \frac{\ln(1 + y_1/a_n)}{\ln a_n} - \frac{\ln(y_1 + a_m)}{\ln a_n} \right.$$
$$\left. + \frac{2\tau(n - 1/2) \ln(1 + y_1/a_n)}{(n \ln a_n/a_n)^2} + \frac{2\tau(m - 1/2) \ln(1 + y_1/a_m)}{(n \ln a_n/a_n)^2} \right) + c_4.$$

Lemma 3.2.3. *Under the hypotheses of Theorem* 3.2.1, *we have*

(1) $\lim_{n \to \infty} (\ln(1 + y_1/a_n)/\ln a_n) = 0$,
(2) $\lim_{n \to \infty} (\ln(y_1 + a_m)/\ln a_n) = 0$,
(3) $\lim_{n \to \infty} (2\tau(n - 1/2) \ln(1 + y_1/a_n)/(n \ln a_n/a_n)^2) = 0$,
(4) $\lim_{n \to \infty} (2\tau(m - 1/2) \ln(1 + y_1/a_m)/(n \ln a_n/a_n)^2) = 0$.

Lemma 3.2.3 implies that there exists a constant c_2, sufficiently close to 1, such that for all n large enough we have

$$\min_{y \geqslant R} I(y_1(n)) \leqslant -\frac{c_2}{2\tau} \left(\frac{n \ln a_n}{a_n} \right)^2 + c_4. \tag{3.2.4}$$

By substituting (3.2.4) into (3.2.3) and taking into account all the changes of variables, we finally obtain that

$$\left\| u(x, t) - \sum_{i=0}^{1} P_n^i(t, \sqrt{A})\, f_i(x) \right\| \leqslant c_1 \exp\left\{ -\frac{c_2 R^2}{t} \left(\frac{n \ln a_n}{a_n} \right)^2 \right\}.$$

The theorem is proved.

Using the approximation theory by S. N. Bernstein, on the basis of the convergence rate of polynomial approximations one can determine the smoothness class of the solution $u(x)$ (see also [3]). With respect to Eqs. (0.2.1)–(0.2.3) with quasianalytical coefficients, this will be studied in our paper to follow.

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